

DUAL FLOWS IN HYPERBOLIC SPACE AND DE SITTER SPACE

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ABSTRACT. We consider contracting flows in $(n+1)$ -dimensional hyperbolic space and expanding flows in $(n+1)$ -dimensional de Sitter space. When the flow hypersurfaces are strictly convex we relate the contracting hypersurfaces and the expanding hypersurfaces by the Gauß map. The contracting hypersurfaces shrink to a point x_0 in finite time while the expanding hypersurfaces converge to the maximal slice $\{\tau = 0\}$. After rescaling, by the same scale factor, the rescaled contracting hypersurfaces converge to a unit geodesic sphere, while the rescaled expanding hypersurfaces converge to slice $\{\tau = -1\}$ exponential fast in $C^\infty(\mathbb{S}^n)$.

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1. INTRODUCTION

In a recent paper [7] a pair of dual flows was considered in \mathbb{S}^{n+1} . The one flow is the contracting flow

$$(1.1) \quad \dot{x} = -F\nu,$$

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while the other is an expanding flow

$$(1.2) \quad \dot{x} = \tilde{F}^{-1}\nu,$$

where $F \in C^\infty(\Gamma_+)$ and \tilde{F} is its inverse

$$(1.3) \quad \tilde{F}(\kappa_i) = \frac{1}{F(\kappa_i^{-1})}.$$

There is a Gauß map for the pair $(\mathbb{S}^{n+1}, \mathbb{S}^{n+1})$, which maps closed, strictly convex hypersurfaces M to their polar sets \tilde{M} , cf. [5, Chapter 9]. Gerhard [7] proved, that the flow hypersurfaces of (1.1) and (1.2) are polar sets of each other, if the initial hypersurface have this property. Under the assumption that F is symmetric, monotone, positive, homogeneous of degree 1, F strictly concave (cf. 3.1) and \tilde{F} concave, it is proved in [7] that the contracting flows contract to a round point and the expanding flows converge to an equator such that after appropriate rescaling, both flows converge to a geodesic sphere exponential fast.

The Gauß map exists also for the pair (\mathbb{H}^{n+1}, N) , where \mathbb{H}^{n+1} is the $(n+1)$ -dimensional hyperbolic space and N is the $(n+1)$ -dimensional de Sitter space, cf. [5, Chapter 10]. We prove in this work similar results as in [7] by using this duality. Let $M(t)$ resp. $\tilde{M}(t)$ be solutions of the contracting flows

$$(1.4) \quad \dot{x} = -F\nu$$

in \mathbb{H}^{n+1} resp. the dual flows

$$(1.5) \quad \dot{x} = -\tilde{F}^{-1}\nu$$

in N , where \tilde{F} is the inverse of F defined by (1.3). We impose the following assumptions.

1.1. Assumption. Let $F \in C^\infty(\Gamma_+)$ be a symmetric, monotone, 1-homogeneous and concave curvature function satisfying the normalization

$$(1.6) \quad F(1, \dots, 1) = 1.$$

We assume further, either

- (1) F is concave and \tilde{F} is concave and the initial hypersurface M_0 is horoconvex (i.e. all principal curvatures $\kappa_i \geq 1$),
- or
- (2) \tilde{F} is convex and M_0 is strictly convex.

We now state our main results

1.2. Theorem. *We consider curvature flows (1.4) and (1.5) under assumption 1.1 with initial smooth hypersurfaces M_0 and \tilde{M}_0 , where \tilde{M}_0 is the polar hypersurface of M_0 . Then the both flows exist on the maximal time interval $[0, T^*)$ with finite T^* . The hypersurfaces $\tilde{M}(t)$ are the polar hypersurfaces of $M(t)$ and vice versa during the evolution. The contracting flow*

hypersurfaces in \mathbb{H}^{n+1} shrink to a point x_0 while the expanding flow hypersurfaces in N converge to a totally geodesic hypersurface which is isometric to \mathbb{S}^n . We may assume the point x_0 is the Beltrami point by applying an isometry such that the hypersurfaces of the expanding flow are all contained in N_- and converge to the coordinate slice $\{\tau = 0\}$.

Viewing \mathbb{H}^{n+1} and N as submanifolds of $\mathbb{R}^{n+1,1}$ and by introducing polar coordinates in the Euclidean part of $\mathbb{R}^{n+1,1}$ centered in $(0, \dots, 0) \in \mathbb{R}^{n+1}$, we can write flow hypersurfaces in \mathbb{H}^{n+1} resp. N as graphs of functions u resp. u^* over \mathbb{S}^n . Let $\Theta = \Theta(t, T^*)$ be the solution of (1.4) with spherical initial hypersurface and existence interval $[0, T^*)$. Then the rescaled functions

$$(1.7) \quad \tilde{u} = u\Theta^{-1}$$

and

$$(1.8) \quad w = u^*\Theta^{-1}$$

are uniformly bounded in $C^\infty(\mathbb{S}^n)$. The rescaled principal curvatures $\kappa_i\Theta$ as well as $\tilde{\kappa}_i\Theta^{-1}$ are uniformly positiv, where $\tilde{\kappa}_i$ are the principal curvatures of $\tilde{M}(t)$.

If the curvature function F is further strictly concave or $F = \frac{1}{n}H$, then the rescaled functions (1.7) resp. (1.8) converge to the constant functions 1 resp. -1 in $C^\infty(\mathbb{S}^n)$ exponentially fast.

Let us review some results concerning the contracting flows in \mathbb{H}^{n+1} . Under the assumption that the initial hypersurface is strictly convex and satisfies the condition $\kappa_i H > n$ for each i , Huisken [11] proved that the flow (1.4) with $F = H$ converges in finite time to a round sphere. Andrews [2] proved similar results for a general class of curvature function with argument $\kappa_i - 1$. Makowski [13] proved the contracting flow with a volume preserving term exists for all times and converges to a geodesic sphere exponentially fast. The key ingredient treating the contracting flow is the pinching estimates. Under assumption 1.1 (1) it follows by a similar calculation as in [13], while Gerhardt [8] proved the pinching estimates under assumption 1.1 (2). The elementary symmetric polynomials are defined by

$$(1.9) \quad H_k(\kappa_1, \dots, \kappa_n) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \kappa_{i_1} \dots \kappa_{i_k}, \quad 1 \leq k \leq n.$$

Examples of curvature functions F satisfying assumption 1.1 (1) (up to normalization condition (1.6)) are

- the power means $(\frac{1}{n} \sum_i \kappa_i^r)^{1/r}$ for $|r| \leq 1$,
- $\sigma_k = H_k^{1/k}$ for $1 \leq k \leq n$,
- the inverse $\tilde{\sigma}_k$ of σ_k for $1 \leq k \leq n$,
- $(H_k/H_l)^{1/(k-l)}$ for $0 \leq l < k \leq n$,
- $H_n^{\alpha_n} H_{n-1}^{\alpha_{n-1} - \alpha_n} \dots H_2^{\alpha_2 - \alpha_3} H_1^{\alpha_1 - \alpha_2}$ for $\alpha_i \geq 0$ and $\sum_i \alpha_i = 1$.

For a proof see [3, Chapter 2]. Moreover, the curvature functions in the above list are all strictly concave with exception of the mean curvature (cf. Section 3)

Examples of convex curvature functions \tilde{F} , which is used in assumption 1.1 (2) (up to normalization condition (1.6)) are (cf. [5, Remark 2.2.13])

- the mean curvature H ,
 - the length of the second fundamental form $|A| = (\sum_i \kappa_i^2)^{1/2}$,
 - the complete symmetric functions
- $$\gamma_k(\kappa_1, \dots, \kappa_n) = \left(\sum_{|\alpha|=k} \kappa_1^{\alpha_1} \kappa_2^{\alpha_2} \dots \kappa_n^{\alpha_n} \right)^{1/k} \text{ for } 1 \leq k \leq n.$$

Note that for convex \tilde{F} under assumption 1.1 (2), F is of class (K) and homogeneous of degree 1, hence strictly concave. (cf. [5, Definition 2.2.1, Lemma 2.2.12, 2.2.14], [7, Lemma 3.6])

2. SETTING AND GENERAL FACTS

We now review some general facts about hypersurfaces from [5, Chapter 1]. Let N be a $(n+1)$ -dimensional semi-Riemannian manifold and M be a hypersurface in N . Geometric quantities in N will be denoted by $(\bar{g}_{\alpha\beta}), (\bar{R}_{\alpha\beta\gamma\delta})$, etc., where greek indices range from 0 to n . Quantities in M will be denoted by $(g_{ij}), (h_{ij})$ etc., where latin indices range from 1 to n . Generic coordinate systems in N resp. M will be denoted by (x^α) resp. (ξ^i) .

Covariant differentiation will usually be denoted by indices, only if ambiguities are possible, by a semicolon, e.g. $h_{ij;k}$.

Let $x : M \hookrightarrow N$ be a spacelike hypersurface (i.e. the induced metric is Riemannian) with a differentiable normal ν , which is always supposed to be normalized, and (h_{ij}) be the second fundamental form, and set $\sigma = \langle \nu, \nu \rangle$. We have the *Gauß formula*

$$(2.1) \quad x_{ij}^\alpha = -\sigma h_{ij} \nu^\alpha,$$

the *Weingarten equation*

$$(2.2) \quad \nu_i^\alpha = h_i^k x_k^\alpha,$$

the *Codazzi equation*

$$(2.3) \quad h_{ij;k} - h_{ik;j} = \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta x_j^\gamma x_k^\delta,$$

and the *Gauß equation*

$$(2.4) \quad R_{ijkl} = \sigma \{ h_{ik} h_{jl} - h_{il} h_{jk} \} + \bar{R}_{\alpha\beta\gamma\delta} x_i^\alpha x_j^\beta x_k^\gamma x_l^\delta.$$

Let us review some properties of \mathbb{H}^{n+1} and N , cf. [5, Section 10.2]. We label the coordinates in the $(n+2)$ -dimensional Minkowski space $\mathbb{R}^{n+1,1}$ as $x = (x^a), 0 \leq a \leq n+1$, where x^0 is the time function. Recall that the hyperbolic space \mathbb{H}^{n+1} and de Sitter space N are the subspaces of $\mathbb{R}^{n+1,1}$ defined by

$$(2.5) \quad \mathbb{H}^{n+1} = \{x \in \mathbb{R}^{n+1,1} : \langle x, x \rangle = -1, x^0 > 0\},$$

$$(2.6) \quad N = \{x \in \mathbb{R}^{n+1,1} : \langle x, x \rangle = 1\}.$$

Introduce polar coordinates in the Euclidean part of $\mathbb{R}^{n+1,1}$ centered in $(0, \dots, 0) \in \mathbb{R}^{n+1}$ such that the metric in $\mathbb{R}^{n+1,1}$ is expressed as

$$(2.7) \quad d\bar{s}^2 = -dx^{0^2} + dr^2 + r^2 \sigma_{ij} d\xi^i d\xi^j,$$

where σ_{ij} is the spherical metric.

By viewing \mathbb{H}^{n+1} as

$$(2.8) \quad \mathbb{H}^{n+1} = \{(x^0, r, \xi^i) : r = \sqrt{|x^0|^2 - 1}, x^0 > 0, \xi \in \mathbb{S}^n\},$$

and by setting

$$(2.9) \quad \varrho = \operatorname{arccosh} x^0,$$

\mathbb{H}^{n+1} has coordinates (ϱ, ξ^i) and the metric

$$(2.10) \quad d\bar{s}_{\mathbb{H}^{n+1}}^2 = d\varrho^2 + \sinh^2 \varrho \sigma_{ij} d\xi^i d\xi^j.$$

Similarly,

$$(2.11) \quad N = \{(x^0, r, \xi^i) : r = \sqrt{1 + |x^0|^2}, x^0 \in \mathbb{R}, \xi \in \mathbb{S}^n\},$$

and by setting the eigentime

$$(2.12) \quad \tau = \operatorname{arcsinh} x^0,$$

N has coordinates (τ, ξ^i) and the metric

$$(2.13) \quad d\bar{s}_N^2 = -d\tau^2 + \cosh^2 \tau \sigma_{ij} d\xi^i d\xi^j.$$

3. STRICTLY CONCAVE CURVATURE FUNCTIONS

For $\xi, \kappa \in \mathbb{R}^n$, we write $\xi \sim \kappa$, if there is $\lambda \in \mathbb{R}$ such that $\xi = \lambda \kappa$.

3.1. Definition. Let $F \in C^2(\Gamma)$ be a symmetric, monotone, 1-homogeneous and concave curvature function. We call F strictly concave (in non-radial directions), if

$$(3.1) \quad F_{ij} \xi^i \xi^j < 0 \quad \forall \xi \not\sim \kappa \text{ and } \xi \neq 0,$$

or equivalently, if the multiplicity of the zero eigenvalue for $D^2 F(\kappa)$ is one for all $\kappa \in \Gamma$.

Note since F is homogeneous of degree 1, $\kappa \in \Gamma$ is an eigenvector of $D^2 F(\kappa)$ with zero eigenvalue. In [7, Chapter 3] it is proved that σ_k , $2 \leq k \leq n$ and the inverses $\tilde{\sigma}_k$ of σ_k , $1 \leq k \leq n$ are strictly concave. In [12, Chapter 2] it is proved that $Q_k = H_{k+1}/H_k$, $1 \leq k \leq n-1$ are strictly concave in Γ_+ . We consider the rest of the concave and inverse concave curvature functions listed on page 3.

3.2. Lemma. *The curvature functions*

$$(3.2) \quad F = \left(\frac{1}{n} \sum_i \kappa_i^r \right)^{1/r} \quad -1 \leq r < 1$$

are strictly concave in Γ_+ .

Proof. Note that F converges locally uniformly to $\sigma_n = (\kappa_1 \cdots \kappa_n)^{1/n}$ as $r \rightarrow 0$ and σ_n is strictly concave. Furthermore, for $-1 \leq r < 1$ and $r \neq 0$,

$$(3.3) \quad \frac{\partial F}{\partial \kappa^i} = n^{-1/r} \left(\sum_l \kappa_l^r \right)^{\frac{1}{r}-1} \kappa_i^{r-1},$$

$$(3.4) \quad \frac{\partial^2 F}{\partial \kappa^i \partial \kappa^j} = n^{-1/r} (1-r) \left(\sum_l \kappa_l^r \right)^{\frac{1}{r}-2} \kappa_i^{r-2} (\kappa_i \kappa_j^{r-1} - \sum_l \kappa_l^r \delta_{ij}).$$

Consider η such that $F_{ij}\eta^j = 0$. Since $r \neq 1$,

$$(3.5) \quad \eta_i = \left(\sum_l \kappa_l^r \right)^{-1} \kappa_j^{r-1} \eta^j \kappa_i.$$

Knowing that F is concave for $|r| \leq 1$ we conclude that F is strictly concave for $-1 \leq r < 1$. \square

3.3. Lemma. *Let f^α be concave in Γ_+ for all $1 \leq \alpha \leq k$ and strictly concave in Γ_+ for at least one index in $1 \leq \alpha \leq k$. Let φ be strictly monotone increasing and concave in Γ_+ , then*

$$(3.6) \quad F(\kappa_1, \dots, \kappa_n) = \varphi(f^1(\kappa_1, \dots, \kappa_n), \dots, f^k(\kappa_1, \dots, \kappa_n))$$

is strictly concave in Γ_+ .

Proof. Let $0 \neq \xi \in \mathbb{R}^n$ and $\xi \not\sim \kappa$, then

$$(3.7) \quad F_{ij}\xi^i \xi^j = \varphi_\alpha f_{ij}^\alpha \xi^i \xi^j + \varphi_{\alpha\beta} f_i^\alpha f_j^\beta \xi^i \xi^j < 0,$$

since by assumption

$$(3.8) \quad \varphi_\alpha > 0, \quad \varphi_{\alpha\beta} \leq 0, \quad f_{ij}^\alpha \xi^i \xi^j \leq 0$$

and

$$(3.9) \quad f_{ij}^\alpha \xi^i \xi^j < 0 \text{ for at least one } 1 \leq \alpha \leq k.$$

\square

Note that the weighted geometric mean

$$(3.10) \quad \varphi(f^1, \dots, f^k) = (f^1)^{\alpha_1} \cdots (f^k)^{\alpha_k} \quad \text{with } \sum_i \alpha_i = 1$$

is a strictly monotone increasing and concave function. Knowing that $H_{k+1}/H_k, 1 \leq k \leq n-1$ are strictly concave in Γ_+ , we conclude that

$$(3.11) \quad (H_k/H_l)^{1/(k-l)} = (H_{l+1}/H_l)^{1/(k-l)} \cdots (H_k/H_{k-1})^{1/(k-l)} \quad 0 \leq l < k \leq n$$

and

$$(3.12) \quad H_n^{\alpha_n} H_{n-1}^{\alpha_{n-1}-\alpha_n} \cdots H_2^{\alpha_2-\alpha_3} H_1^{\alpha_1-\alpha_2} = \left(\frac{H_1}{H_0} \right)^{\alpha_1} \left(\frac{H_2}{H_1} \right)^{\alpha_2} \cdots \left(\frac{H_n}{H_{n-1}} \right)^{\alpha_n}$$

with $\alpha_i \geq 0$, $\sum_i \alpha_i = 1$ and $\alpha_1 \neq 1$ are strictly concave in Γ_+ .

4. POLAR SETS AND DUAL FLOWS

We state some facts about Gauß maps for (\mathbb{H}^{n+1}, N) , cf. [5, Section 10.4].

4.1. Theorem. *Let $x : M_0 \rightarrow M \subset \mathbb{H}^{n+1}$ be a closed, connected, strictly convex hypersurface. Consider M as a codimension 2 immersed submanifold in $\mathbb{R}^{n+1,1}$ such that*

$$(4.1) \quad x_{ij} = g_{ij}x - h_{ij}\tilde{x},$$

where $\tilde{x} \in T_x(\mathbb{R}^{n+1,1})$ is the representation of the exterior normal vector $\nu = (\nu^\alpha)$ of M in $T_x(\mathbb{H}^{n+1})$. Then the Gauß map

$$(4.2) \quad \tilde{x} : M_0 \rightarrow N$$

is the embedding of a closed, spacelike, achronal, strictly convex hypersurface $\tilde{M} \subset N$. Viewing \tilde{M} as a codimension 2 submanifold in $\mathbb{R}^{n+1,1}$, its Gaussian formula is

$$(4.3) \quad \tilde{x}_{ij} = -\tilde{g}_{ij}\tilde{x} + \tilde{h}_{ij}x,$$

where $\tilde{g}_{ij}, \tilde{h}_{ij}$ are the metric and second fundamental form of \tilde{M} and x is the embedding of M which also represents the future directed normal vector of \tilde{M} . The second fundamental form \tilde{h}_{ij} is defined with respect to the future directed normal vector, where the time orientation of N is inherited from $\mathbb{R}^{n+1,1}$. Furthermore, there holds

$$(4.4) \quad \tilde{h}_{ij} = h_{ij},$$

$$(4.5) \quad \tilde{\kappa}_i = \kappa_i^{-1}.$$

□

We prove in the following that the duality is also valid in case of curvature flows.

4.2. Lemma. *Let $\Phi \in C^\infty(\mathbb{R}_+)$ be strictly monotone, $\dot{\Phi} > 0$, and let $F \in C^\infty(\Gamma_+)$ be a symmetric, monotone, 1-homogeneous curvature function such that $F|_{\Gamma_+} > 0$ and such that the flows*

$$(4.6) \quad \dot{x} = -\Phi(F)\nu$$

in \mathbb{H}^{n+1} resp.

$$(4.7) \quad \dot{\tilde{x}} = -\Phi(\tilde{F}^{-1})\tilde{\nu}$$

in N with initial strictly convex hypersurfaces M_0 resp. \tilde{M}_0 exist on maximal time intervals $[0, T^*)$ resp. $[0, \tilde{T}^*)$, where ν and $\tilde{\nu}$ are the exterior normal resp. past directed normal. The flow hypersurfaces are then strictly convex. Let $M(t)$ resp. $\tilde{M}(t)$ be the corresponding flow hypersurfaces, then $T^* = \tilde{T}^*$ and $M(t) = \tilde{M}(t)$ for all $t \in [0, T^*)$.

Proof. The arguments are similar to those in [7, Section 4] with combination with the results from [5, Section 10.4]. Since there holds

$$(4.8) \quad \langle x, x \rangle = 1, \langle \dot{x}, x \rangle = 0, \langle x_j, x \rangle = 0, \langle \tilde{x}, x \rangle = 0,$$

(see [5, Lemma 10.4.1] for the last identity) we can consider the flow (4.6) as flow in $\mathbb{R}^{n+1,1}$

$$(4.9) \quad \dot{x} = -\Phi \tilde{x},$$

and we have the decomposition

$$(4.10) \quad T_x(\mathbb{R}^{n+1,1}) = T_x(\mathbb{H}^{n+1}) \oplus \langle x \rangle.$$

Furthermore, we conclude from

$$(4.11) \quad \langle \dot{\tilde{x}}, x_j \rangle = \Phi_j, \langle \dot{\tilde{x}}, \tilde{x} \rangle = 0, \langle \dot{\tilde{x}}, x \rangle = \Phi,$$

from the Weingarten equation (see [5, Lemma 10.4.3, 10.4.4])

$$(4.12) \quad x_j = \tilde{h}_j^k \tilde{x}_k,$$

and from (4.10) that

$$(4.13) \quad \dot{\tilde{x}} = \Phi x + \Phi^m x_m = \Phi x + \Phi^m \tilde{h}_m^k \tilde{x}_k,$$

where

$$(4.14) \quad \Phi^m = g^{mj} \Phi_j,$$

and the second fundamental form \tilde{h}_{ij} is defined with respect to the future directed normal vector $\tilde{\nu}$. The corresponding flow equation in N has the form

$$(4.15) \quad \dot{\tilde{x}} = \Phi \tilde{\nu} + \Phi^m \tilde{h}_m^k \tilde{x}_k.$$

Let $t_0 \in [0, T^*)$ and introduce polar coordinates in the Euclidean part of the Minkowski space as well as an eigentime coordinate system in N as in Section 2. For ϵ small and $t_0 < t < t_0 + \epsilon$, $\tilde{M}(t)$ can be written as graph over \mathbb{S}^n

$$(4.16) \quad \tilde{M}(t) = \text{graph } \tilde{u}|_{\mathbb{S}^n},$$

and we obtain the scalar flow equation

$$(4.17) \quad \frac{d\tilde{u}}{dt} = \Phi \tilde{\nu}^{-1} + \Phi^m \tilde{h}_m^k \tilde{u}_k,$$

where

$$(4.18) \quad \tilde{\nu}^2 = 1 - |D\tilde{u}|^2 = 1 - \frac{1}{\cosh^2 \tilde{u}} \sigma^{ij} \tilde{u}_i \tilde{u}_j.$$

Note that $\tilde{\nu}$ in (4.15) is the future directed normal

$$(4.19) \quad (\tilde{\nu}^\alpha) = \tilde{\nu}^{-1} (1, \tilde{u}^i),$$

where

$$(4.20) \quad \tilde{u}^i = \frac{1}{\cosh^2 \tilde{u}} \sigma^{ij} \tilde{u}_j.$$

Thus it holds in view of (4.15)

$$\begin{aligned}
 (4.21) \quad \frac{\partial \tilde{u}}{\partial t} &= \frac{d\tilde{u}}{dt} - \tilde{u}_i \dot{x}^i \\
 &= \Phi \tilde{v}^{-1} + \Phi^m h_m^k \tilde{u}_k - \Phi \tilde{v}^{-1} |D\tilde{u}|^2 - \Phi^m \tilde{h}_m^k \delta_k^i \tilde{u}_i \\
 &= \Phi \tilde{v}.
 \end{aligned}$$

This is exactly the scalar curvature equation of the flow equation

$$(4.22) \quad \dot{x} = -\Phi \tilde{\nu},$$

where $\tilde{\nu}$ in (4.22) is the future directed normal and

$$(4.23) \quad \Phi = \Phi(F) = \Phi(\tilde{F}^{-1}).$$

Now \tilde{h}_{ij} in N is defined with respect to the future directed normal. By adapting the convention in [5, p.307] we switch the light cone in N and by defining $\tau = -\operatorname{arcsinh} x^0$ in (2.12) we still derive the flow (4.22) in N , where $\tilde{\nu}$ is now the past directed normal and the second fundamental form is defined with respect to this normal. The rest of the proof is identical to [7, Theorem 4.2]. \square

From now we shall employ this duality by choosing

$$(4.24) \quad \Phi(r) = r.$$

Note that the expanding flows in \mathbb{H}^{n+1} was already considered in [6] and its non-scale-invariant version in [14].

5. PINCHING ESTIMATES

We consider the contracting flow

$$\begin{aligned}
 (5.1) \quad \dot{x} &= -F\nu, \\
 x(0) &= M_0
 \end{aligned}$$

in \mathbb{H}^{n+1} with initial smooth and strictly convex hypersurfaces M_0 , where ν is the exterior normal vector.

Under the assumptions of Theorem 1.2 the curvature flow (5.1) exists on a maximal time interval $[0, T^*)$, $0 < T^* \leq \infty$, cf. [5, Theorem 2.5.19, Lemma 2.6.1].

5.1. Theorem. *Let $M(t)$ be a solution of the flow (5.1) in \mathbb{H}^{n+1} . If the initial hypersurface M_0 in \mathbb{H}^{n+1} satisfies*

$$(5.2) \quad \kappa_i > 1,$$

then this condition will also be satisfied by the flow hypersurfaces $M(t)$ during the evolution.

Proof. The tensor

$$(5.3) \quad S_{ij} = h_{ij} - g_{ij}$$

satisfies the equation

$$(5.4) \quad \begin{aligned} \dot{S}_{ij} - F^{kl} S_{ij;kl} &= F^{kl} h_{rk} h_l^r h_{ij} - 2F h_i^k h_{kj} \\ &\quad + K_N \{2F g_{ij} - F^{kl} g_{kl} h_{ij}\} + 2F h_{ij} + F^{kl,rs} h_{kl;i} h_{rs;j} \\ &\equiv N_{ij} + \tilde{N}_{ij}, \end{aligned}$$

where $\tilde{N}_{ij} = F^{kl,rs} h_{kl;i} h_{rs;j}$. At every point where $h_{ij} \eta^j = \eta_i$ there holds

$$(5.5) \quad N_{ij} \eta^i \eta^j = \{F^{kl} h_{rk} h_l^r - 2F + F^{kl} g_{kl}\} |\eta|^2 \geq 0.$$

It was proved in [3, Theorem 3.3, Lemma 4.4] that

$$(5.6) \quad \tilde{N}_{ij} \eta^i \eta^j + \sup_{\Gamma} 2F^{kl} \{2\Gamma_l^r S_{ir;k} \eta^i - \Gamma_k^r \Gamma_l^s S_{rs}\} \geq 0,$$

where only the inverse concavity of F was used. Andrews' maximum principle in [3, Theorem 3.2] implies that $S_{ij} > 0$ during the evolution. \square

In the next step we use a constant rank theorem to allow the condition $\kappa_i \geq 1$ in the proof of the succeeding Lemma 5.4.

5.2. Lemma. *Let $M(t)$ be a solution of the flow (5.1) in \mathbb{H}^{n+1} and assume that the tensor S_{ij} satisfies $S_{ij} \geq 0$ on the hypersurfaces $M(t)$ for $t \in [0, T^*)$, then S_{ij} is of constant rank $l(t)$ for every $t \in (0, T^*)$.*

Proof. The proof is similar to those in [15, Theorem 3.2], where the main part is based on the computation in [4, Theorem 3.2]. For $\epsilon > 0$, let

$$(5.7) \quad W_{ij} = S_{ij} + \epsilon g_{ij}.$$

Let $l(t)$ be the minimal rank of $S_{ij}(t)$. For a fixed $t_0 \in (0, T^*)$, let $x_0 \in M(t_0)$ be the point such that $S_{ij}(t_0, \xi)$ attains its minimal rank at x_0 . Set

$$(5.8) \quad \phi(t, \xi) = H_{l+1}(W_{ij}(t, \xi)) + \frac{H_{l+2}(W_{ij}(t, \xi))}{H_{l+1}(W_{ij}(t, \xi))},$$

where H_l is the elementary symmetric polynomials of eigenvalues of W_{ij} , homogeneous of order l . A direct computation shows

$$(5.9) \quad \begin{aligned} F^{kl} W_{ij;kl} - \dot{W}_{ij} &= -F^{kl} h_{rk} h_l^r W_{ij} - F^{kl} g_{kl} W_{ij} + 2F h_i^k W_{kj} \\ &\quad - F^{kl,rs} W_{kl;i} W_{rs;j} + 2F \epsilon g_{ij} \\ &\quad - (1 - \epsilon) \{F^{kl} h_{rk} h_l^r - 2F + F^{kl} g_{kl}\} g_{ij}. \end{aligned}$$

As in [4], we consider a neighborhood $(t_0 - \delta, t_0] \times \mathcal{O}$ around (t_0, ξ_0) . We use the notation $h = O(f)$ if $|h(\xi)| \leq C f(\xi)$ for every $(t, \xi) \in (t_0 - \delta, t_0] \times \mathcal{O}$, where C is a constant, depending on the $C^{1,1}$ norm of the second fundamental form on $(t_0 - \delta, t_0] \times \mathcal{O}$, but independent of ϵ . It was proved in [4, Corollary 2.2] that ϕ is in $C^{1,1}$. And as in [4], let $G = \{n - l + 1, n - l + 2, \dots, n\}$

and $B = \{1, \dots, n-l\}$. We choose the coordinates such that $h_{ij} = \kappa_i \delta_{ij}$ and $g_{ij} = \delta_{ij}$. In view of [4, (3.14)], in such coordinates ϕ^{ij} is up to $O(\phi)$ non-negative in \mathcal{O} and we have

$$(5.10) \quad \begin{aligned} F^{kl} \phi_{;kl} - \dot{\phi} &\leq \phi^{ij} \{-F^{kl} h_{rk} h_l^r W_{ij} - F^{kl} g_{kl} W_{ij} + 2F h_i^k W_{kj} \\ &\quad + 2F \epsilon g_{ij} - F^{kl,rs} W_{kl;i} W_{rs;j}\} + F^{kl} \phi^{ij,rs} W_{ij;k} W_{rs;l} + O(\phi). \end{aligned}$$

We can choose \mathcal{O} small enough, such that $\epsilon = O(\phi)$ as in [4, (3.8)]. It was proved in [4, (3.14)] that $\phi^{ii} = O(\phi)$ for $i \in G$ and since $W_{ii} \leq \phi$ for $i \in B$, we infer that

$$(5.11) \quad F^{kl} \phi_{;kl} - \dot{\phi} \leq -\phi^{ij} F^{kl,rs} W_{kl;i} W_{rs;j} + F^{kl} \phi^{ij,rs} W_{ij;k} W_{rs;l} + O(\phi).$$

Using the inverse concavity of F and proceed as in [4, Theorem 3.2], we conclude

$$(5.12) \quad F^{kl} \phi_{;kl} - \dot{\phi} \leq C\{\phi + |D\phi|\},$$

where C is a constant independent of ϵ and ϕ . Taking $\epsilon \rightarrow 0$, the strong maximum principle for parabolic equations yields

$$(5.13) \quad H_{l(t_0)+1}(S_{ij}(t, \xi)) \equiv 0 \quad \forall (t, \xi) \in (t_0 - \delta, t_0] \times \mathcal{O}.$$

Since $M(t_0)$ is a closed hypersurface, $S_{ij}(t_0, \xi)$ is of constant rank $l(t_0)$ on $M(t_0)$. \square

Note that the proof of the Lemma 5.1 implies, if the initial hypersurface satisfies $\kappa_i \geq 1$, then this condition remains true during the evolution. Furthermore, every closed hypersurface in \mathbb{H}^{n+1} contains a point on which holds $\kappa_i > 1$. Thus we conclude

5.3. Corollary. *Let $M(t)$ be a solution of the flow (5.1) in \mathbb{H}^{n+1} . If the initial hypersurface M_0 in \mathbb{H}^{n+1} satisfies $\kappa_i \geq 1$, then $\kappa_i > 1$ for every $t \in (0, T^*)$.*

5.4. Lemma. *Let $M(t)$ be a solution of the flow (5.1) in \mathbb{H}^{n+1} under assumption 1.1 (1), then there exists a uniform positive constant $\epsilon > 0$ such that*

$$(5.14) \quad \kappa_1 \geq \epsilon \kappa_n$$

during the evolution, where the principal curvatures are labeled as

$$(5.15) \quad \kappa_1 \leq \dots \leq \kappa_n.$$

Proof. The proof is similar to [13, Lemma 4.2]. By Replacing M_0 by $M(t_0)$ for a $t_0 \in (0, T^*)$ as initial hypersurface, we can assume that $\kappa_i > 1$ on M_0 . Let F be a concave and inverse concave curvature function, then

$$(5.16) \quad T_{ij} = h_{ij} - g_{ij} - \epsilon(H - n)g_{ij}$$

satisfies the equation

$$\begin{aligned}
(5.17) \quad \dot{T}_{ij} - F^{kl}T_{ij;kl} &= F^{kl}h_{rk}h_l^r\{h_{ij} - \epsilon H g_{ij}\} - 2Fh_i^k\{h_{kj} - \epsilon H g_{kj}\} \\
&\quad + 2K_N F g_{ij} - 2n\epsilon K_N F g_{ij} - K_N F^{kl}g_{kl}\{h_{ij} - \epsilon H g_{ij}\} \\
&\quad - 2F(\epsilon n - 1)h_{ij} + F^{kl,rs}h_{kl;i}h_{rs;j} - \epsilon F^{kl,rs}h_{kl;p}h_{rs;q}g^{pq}g_{ij} \\
&\equiv N_{ij} + \tilde{N}_{ij},
\end{aligned}$$

where $\tilde{N}_{ij} = F^{kl,rs}h_{kl;i}h_{rs;j} - \epsilon F^{kl,rs}h_{kl;p}h_{rs;q}g^{pq}g_{ij}$.

At every point where $T_{ij}\eta^j = 0$ there holds

$$\begin{aligned}
(5.18) \quad N_{ij}\eta^i\eta^j &= F^{kl}h_{rk}h_l^r(1 - \epsilon n)|\eta|^2 + 2Fh_{ij}(\epsilon n - 1)\eta^i\eta^j \\
&\quad + \{F^{kl}g_{kl} - 2F\}(1 - \epsilon n)|\eta|^2 - 2F(\epsilon n - 1)h_{ij}\eta^i\eta^j \\
&= (1 - \epsilon n)\sum_i F_i(\kappa_i^2 - 2\kappa_i + 1)|\eta|^2 \geq 0.
\end{aligned}$$

It is proved in [1, Theorem 4.1] (see also the modification in [13, Theorem B.2]) that

$$(5.19) \quad \tilde{N}_{ij}\eta^i\eta^j + \sup_{\Gamma} 2F^{kl}\{2\Gamma_l^r T_{ir;k}\eta^i - \Gamma_k^r \Gamma_l^s T_{rs}\} \geq 0,$$

We can choose $\epsilon > 0$ sufficiently small, such that $T_{ij} \geq 0$ on M_0 , then the Andrews' maximum principle [3, Theorem 3.2] implies $T_{ij} \geq 0$ and hence

$$(5.20) \quad \kappa_1 - 1 \geq \epsilon(H - n)$$

during the evolution. \square

The following pinching results is due to Gerhard. By using [8, Theorem 1.1] and the duality result Lemma 4.2 we obtain

5.5. Theorem. *Let $M(t)$ be a solution of the flow (5.1) in \mathbb{H}^{n+1} under the assumption 1.1 (2), then there exists a uniform constant $\epsilon > 0$ such that*

$$(5.21) \quad \kappa_1 \geq \epsilon \kappa_n$$

during the evolution.

6. CONTRACTING FLOWS - CONVERGENCE TO A POINT

Fix a point $p_0 \in \mathbb{H}^{n+1}$, the hyperbolic metric in the geodesic polar coordinates centered at p_0 can be expressed as

$$(6.1) \quad d\bar{s}^2 = dr^2 + \sinh^2 r \sigma_{ij} dx^i dx^j,$$

where σ_{ij} is the canonical metric of \mathbb{S}^n .

Geodesic spheres with center in p_0 are totally umbilic. The induced metric, second fundamental form and the principal curvatures of the coordinate slices $S_r = \{x^0 = r\}$ are given by

$$(6.2) \quad \bar{g}_{ij} = \sinh^2 r \sigma_{ij}, \quad \bar{h}_{ij} = \frac{1}{2} \dot{\bar{g}}_{ij} = \coth r \bar{g}_{ij}, \quad \bar{\kappa}_i = \coth r,$$

respectively. See [5, (1.5.12)].

6.1. Lemma. *Consider (5.1) with initial hypersurface $x(0) = S_{r_0}$, then the corresponding flow exists in a maximal time interval $[0, T^*)$ with T^* finite and will shrink to a point. The flow hypersurfaces $M(t)$ are all geodesic spheres with the same center and their radii $\Theta = \Theta(t)$ solve the ODE*

$$(6.3) \quad \begin{aligned} \dot{\Theta} &= -\coth \Theta, \\ \Theta(0) &= r_0. \end{aligned}$$

Proof. We set

$$(6.4) \quad \begin{aligned} x^0(t, \xi) &= \Theta(t), \\ x^i(t, \xi) &= x^i(0, \xi). \end{aligned}$$

In view of [5, (1.5.7)] the exterior normal of a geodesic sphere is $(1, 0, \dots, 0)$. Using that $F(\bar{h}_j^i) = \coth \Theta$, we see that x in (6.4) solves the flow equation (5.1). Now the solution of (6.3) is given by

$$(6.5) \quad \cosh \Theta = (\cosh r_0) e^{-t}.$$

Thus the spherical flow exists only for a finite time $[0, T^*)$. Note that (6.5) can be rewritten as

$$(6.6) \quad \Theta = \operatorname{arccosh} e^{(T^* - t)}.$$

□

Next we want to prove that the flow (5.1) shrinks to a point. Using the inverse of the Beltrami map, \mathbb{H}^{n+1} is parametrizable over $B_1(0)$ yielding the metric (cf. [5, Section 10.2])

$$(6.7) \quad d\bar{s}^2 = \frac{1}{(1-r^2)^2} dr^2 + \frac{r^2}{1-r^2} \sigma_{ij} d\xi^i d\xi^j.$$

Define the variable ϱ by

$$(6.8) \quad \varrho = \operatorname{arctanh} r = \frac{1}{2}(\log(1+r) - \log(1-r)),$$

then

$$(6.9) \quad d\bar{s}^2 = d\varrho^2 + \sinh^2 \varrho \sigma_{ij} d\xi^i d\xi^j.$$

Let

$$(6.10) \quad d\bar{s}^2 = dr^2 + r^2 \sigma_{ij} d\xi^i d\xi^j$$

be the Euclidean metric over $B_1(0)$. Define

$$(6.11) \quad d\tau = \frac{1}{r\sqrt{1-r^2}} dr, \quad d\tilde{\tau} = r^{-2} dr,$$

we have further

$$(6.12) \quad \begin{aligned} d\tilde{s}^2 &= \frac{r^2}{1-r^2} \{d\tau^2 + \sigma_{ij} d\xi^i d\xi^j\} \equiv e^{2\psi} \{d\tau^2 + \sigma_{ij} d\xi^i d\xi^j\}, \\ d\tilde{s}^2 &= r^2 \{d\tilde{\tau}^2 + \sigma_{ij} d\xi^i d\xi^j\} \equiv e^{2\tilde{\psi}} \{d\tilde{\tau}^2 + \sigma_{ij} d\xi^i d\xi^j\}. \end{aligned}$$

An arbitrary closed, connected, strictly embedded hypersurface $M \subset \mathbb{H}^{n+1}$ bounds a convex body and we can write M as a graph in geodesic polar coordinates.

$$(6.13) \quad M = \text{graph } u = \{\tau = u(x) : x \in \mathbb{S}^n\}.$$

M can also be viewed as a graph \tilde{M} in $B_1(0)$ with respect to the Euclidean metric

$$(6.14) \quad \tilde{M} = \text{graph } \tilde{u} = \{\tilde{\tau} = \tilde{u}(x) : x \in \mathbb{S}^n\}.$$

Writing $\tilde{u} = \varphi(u)$, then there holds (see [5, (10.2.18)])

$$(6.15) \quad \dot{\varphi}^2 = 1 - r^2.$$

The same argument as in [7, Lemma 6.1] yields

6.2. Lemma. *Let $M(t)$ be a solution of (5.1) on a maximal time interval $[0, T^*)$ and represent $M(t)$, for a fixed $t \in [0, T^*)$, as a graph in polar coordinates with center in $x_0 \in \hat{M}(t)$*

$$(6.16) \quad M(t) = \text{graph } u(t, \cdot),$$

then

$$(6.17) \quad \inf_{M(t)} u \leq \Theta(t, T^*) \leq \sup_{M(t)} u,$$

where the solution of the spherical flow $\Theta(t, T^*)$ is given by (6.6). \square

6.3. Lemma. *Let $x_0 \in \hat{M}(t)$ be as above and represent $M(t)$ in Euclidean polar coordinates (6.10), then there exists a constant $c_0 = c_0(M_0) < 1$ such that the estimate*

$$(6.18) \quad r \leq c_0$$

holds for any $t \in [0, T^*)$.

Proof. The argument is similar to those in [7, Lemma 6.3, Remark 6.5]. Looking at the scalar flow equation for a short time interval, we conclude that the convex bodies $\hat{M}(t) \subset \mathbb{H}^{n+1}$ are decreasing with respect to t . Furthermore, \hat{M}_0 is strictly convex. Thus ϱ is uniformly bounded and the claim follows from the relation

$$(6.19) \quad r = \tanh \varrho = 1 - \frac{2}{e^{2\varrho} + 1}.$$

\square

Denote h_{ij} resp. \tilde{h}_{ij} the second fundamental forms and κ_i resp. $\tilde{\kappa}_i$ the principal curvatures of M with respect to the ambient metric $\bar{g}_{\alpha\beta}$ resp. $\tilde{g}_{\alpha\beta}$.

6.4. Lemma. *The principal curvatures $\tilde{\kappa}_i$ of $M(t)$ are pinched, i.e., there exists a uniform constant c such that*

$$(6.20) \quad \tilde{\kappa}_n \leq c\tilde{\kappa}_1,$$

where the $\tilde{\kappa}_i$ are labeled as

$$(6.21) \quad \tilde{\kappa}_1 \leq \cdots \leq \tilde{\kappa}_n.$$

Proof. The h_{ij} and \tilde{h}_{ij} are related through the formula (see [5, (10.2.33)])

$$(6.22) \quad \tilde{h}_{ij}\tilde{v} = (1 - r^2)h_{ij}v,$$

where

$$(6.23) \quad \begin{aligned} v^2 &= 1 + \sigma^{ij}u_iu_j, \\ \tilde{v}^2 &= 1 + \varphi^2\sigma^{ij}u_iu_j. \end{aligned}$$

Because of Lemma 6.3 there exists $0 < \delta < 1$ such that

$$(6.24) \quad r^2 \leq 1 - \delta,$$

and thus

$$(6.25) \quad \delta v^2 \leq \tilde{v}^2 \leq v^2,$$

$$(6.26) \quad \delta h_{ij} \leq \tilde{h}_{ij} \leq \delta^{-1}h_{ij}.$$

Furthermore, there holds

$$(6.27) \quad \begin{aligned} g_{ij} &= \frac{r^2}{1 - r^2} \{u_iu_j + \sigma_{ij}\}, \\ \tilde{g}_{ij} &= r^2 \{\varphi^2 u_iu_j + \sigma_{ij}\}. \end{aligned}$$

and we conclude

$$(6.28) \quad \delta^2 g_{ij} \leq \tilde{g}_{ij} \leq g_{ij}.$$

Now the claim follows from the maximum-minimum principle. \square

For $\hat{M}(t) \subset \mathbb{H}^{n+1}$, the inradius $\rho_-(t)$ and circumradius $\rho_+(t)$ of $\hat{M}(t)$ are defined by

$$(6.29) \quad \begin{aligned} \rho_-(t) &= \sup\{r : B_r(y) \text{ is enclosed by } \hat{M}(t) \text{ for some } y \in \mathbb{H}^{n+1}\}, \\ \rho_+(t) &= \inf\{r : B_r(y) \text{ encloses } \hat{M}(t) \text{ for some } y \in \mathbb{H}^{n+1}\}. \end{aligned}$$

Now, choose $x_0 \in \hat{M}(t)$ to be the center of the inball of $\hat{M}(t) \subset \mathbb{H}^{n+1}$ and let x_0 be the center of the geodesic polar coordinates. Note that the center of the Euclidean inball is also x_0 . Let $\rho_-(t)$ resp. $\rho_+(t)$ be the inradius resp. circumradius of $\hat{M}(t) \subset \mathbb{H}^{n+1}$, and let $\tilde{\rho}_-(t)$ resp. $\tilde{\rho}_+(t)$ be the inradius resp. circumradius of $\hat{M}(t) \subset \mathbb{R}^{n+1}$.

6.5. Lemma. *Let $B_{\rho_-(t)}(x_0) \subset \hat{M}(t)$ be a geodesic inball, then there exist positive constants c and δ , such that*

$$(6.30) \quad \hat{M}(t) \subset B_{4c\rho_-(t)}(x_0) \quad \forall t \in [T^* - \delta, T^*].$$

Proof. The pinching estimates in the Euclidean ambient space (6.20) and [1, Theorem 5.1, Theorem 5.4] imply

$$(6.31) \quad \tilde{\rho}_+(t) \leq c\tilde{\rho}_-(t)$$

with a uniform constant c , hence $\hat{M}(t)$ is contained in the Euclidean ball $B_{\tilde{\rho}}(0)$,

$$(6.32) \quad \hat{M}(t) \subset B_{\tilde{\rho}}(0), \quad \tilde{\rho}(t) = 2c\tilde{\rho}_-(t).$$

Furthermore, we deduce from Lemma 6.2 that

$$(6.33) \quad \inf_{M(t)} \tilde{u} \leq \tilde{\Theta} \leq \sup_{M(t)} \tilde{u},$$

where $M(t) = \text{graph } \tilde{u}$ is a representation of $M(t)$ in Euclidean polar coordinates. We conclude further

$$(6.34) \quad \tilde{\rho}(t) = 2c\tilde{\rho}_-(t) \leq 2c\tilde{\Theta}.$$

Choose now $\delta > 0$ small such that

$$(6.35) \quad 2c\tilde{\Theta}(t, T^*) \leq 1 \quad \forall t \in [T^* - \delta, T^*].$$

Now it holds for

$$(6.36) \quad \rho(t) = \text{arctanh } \tilde{\rho}(t)$$

$$(6.37) \quad \hat{M}(t) \subset B_{\rho(t)}(x_0) \subset \mathbb{H}^{n+1}.$$

Since

$$(6.38) \quad \tilde{\rho}(t) \leq 1,$$

we conclude further

$$(6.39) \quad \tilde{\rho} \leq \rho \leq 2\tilde{\rho}, \quad \tilde{\rho}_- \leq \rho_-.$$

Thus

$$(6.40) \quad \rho \leq 2\tilde{\rho} = 4c\tilde{\rho}_- \leq 4c\rho_-$$

and the claim follows. \square

6.6. Lemma. *During the evolution the flow hypersurfaces $M(t)$ are smooth and uniformly convex satisfying a priori estimates in any compact subinterval $[0, T] \subset [0, T^*)$.*

Proof. Let $0 < T < T^*$ be fixed. From (6.31) and (6.33) we infer

$$(6.41) \quad c\tilde{\Theta}(T, T^*) \leq \tilde{\rho}_-(T).$$

Since

$$(6.42) \quad \Theta(T, T^*) = \text{arctanh } \tilde{\Theta}(T, T^*), \quad \rho_-(T) = \text{arctanh } \tilde{\rho}_-(T),$$

and $\tilde{\rho}_-(T)$, $\tilde{\Theta}(T, T^*)$ are uniformly bounded from above by 1 we infer that

$$(6.43) \quad 0 < \frac{c}{2}\Theta = \frac{c}{2}\text{arctanh } \tilde{\Theta} \leq c\tilde{\Theta} \leq \text{arctanh}(c\tilde{\Theta}) \leq \rho_-(T).$$

Let $x_0 \in \hat{M}(T)$ be the center of an inball and introduce geodesic polar coordinates with center x_0 . This coordinate system will cover the flow in $0 \leq t \leq T$. Writing the flow hypersurfaces as graphs $u(t, \cdot)$ of a function we have

$$(6.44) \quad 0 < c^{-1} \leq u \leq c.$$

And since $M(t)$ are convex,

$$(6.45) \quad v^2 = 1 + \sinh^{-2} u \sigma^{ij} u_i u_j$$

is uniformly bounded. Under assumption 1.1 (1) we have $\kappa_i \geq 1$. And under assumption 1.1 (2) it is proved in [8, Lemma 4.4] that

$$(6.46) \quad \frac{1}{n} \tilde{\kappa}_n \leq \tilde{F} \leq c$$

in N or equivalently, $\kappa_i \geq c$ in \mathbb{H}^{n+1} . The proof of uniform boundedness of κ_i from above is similar to those in [7, Theorem 6.6]. Since F is concave, we may first apply the Krylov-Safonov and then the parabolic Schauder estimates to obtain the desired a priori estimates. \square

In view of Lemma 6.1, 6.2, 6.5 and 6.6, the flow (5.1) shrinks in finite time to a point x_0 .

7. THE RESCALED FLOW

In view of Lemma 6.2 and 6.5 we can choose $\delta > 0$ small and define

$$(7.1) \quad t_\delta = T^* - \delta,$$

such that

$$(7.2) \quad \hat{M}(t_\delta) \subset B_{8c\rho_-(t_\delta)}(x_0) \quad \forall x_0 \in \hat{M}(t_\delta),$$

and

$$(7.3) \quad 8c\rho_-(t_\delta) \leq 8c\Theta(t_\delta, T^*) < 1.$$

Fix now a $t_0 \in (t_\delta, T^*)$ and let $B_{\rho_-(t_0)}(x_0)$ be an inball of $\hat{M}(t_0)$. Choose x_0 to be the center of a geodesic polar coordinate system, then the hypersurfaces $M(t)$ can be written as graphs

$$(7.4) \quad M(t) = \text{graph } u(t, \cdot) \quad \forall t_\delta \leq t \leq t_0,$$

such that

$$(7.5) \quad \rho_-(t_0) \leq u(t_0) \leq u(t) \leq 1.$$

7.1. Lemma. *Let*

$$(7.6) \quad \chi = \frac{v}{\sinh u} \equiv v\eta(r),$$

if $\chi_i = 0$, then $u_i = 0$.

Proof. Note that

$$(7.7) \quad \eta(r) = \frac{1}{\sinh r}$$

solves the equation

$$(7.8) \quad \dot{\eta} = -\frac{\bar{H}}{n}\eta,$$

hence the proof is same as those in [7, Lemma 7.1]. \square

Similar to [7, Lemma 7.2, Corollary 7.3] we obtain

7.2. Lemma. *There exists a uniform constant $c > 0$ such that*

$$(7.9) \quad \Theta(t, T^*)F \leq c \quad \forall t \in [t_\delta, T^*),$$

and that the rescaled principal curvatures $\tilde{\kappa}_i = \kappa_i \Theta$ satisfy

$$(7.10) \quad \tilde{\kappa}_i \leq c \quad \forall t \in [t_\delta, T^*).$$

\square

7.3. Lemma. *Let $t_1 \in [t_\delta, T^*)$ be arbitrary and let $t_2 > t_1$ be such that*

$$(7.11) \quad \Theta(t_2, T^*) = \frac{1}{2}\Theta(t_1, T^*).$$

Let $x_0 \in \hat{M}(t_2)$ be the center of an geodesic inball and introduce polar coordinates around x_0 and write the hypersurface $M(t)$ as graphs

$$(7.12) \quad M(t) = \text{graph } u(t, \cdot).$$

Define ϑ by

$$(7.13) \quad \vartheta(r) = \sinh r,$$

and

$$(7.14) \quad \varphi = \int_{r_2}^u \vartheta^{-1},$$

where $r_2 = \Theta(t_2, T^)$. Then $\varphi(t, \cdot)$ is uniformly bounded in $C^2(\mathbb{S}^n)$ for any $t_1 \leq t \leq t_2$ independent of t_1, t_2 . Furthermore, let Γ_{ij}^k and $\tilde{\Gamma}_{ij}^k$ be the Christoffel symbols of the metrics g_{ij} and σ_{ij} respectively, then the tensor $\Gamma_{ij}^k - \tilde{\Gamma}_{ij}^k$ is also uniformly bounded independent of t_1, t_2 .*

Proof. As in [7, Lemma 7.4], we conclude from Lemma 6.2 and Lemma 6.5 that there exists a uniform constant $c > 1$, independent of t_1, t_2 , such that

$$(7.15) \quad c^{-1}\Theta(t_2, T^*) \leq u(t, \xi) \leq c\Theta(t_2, T^*) \quad \forall t \in [t_1, t_2].$$

Note that

$$(7.16) \quad \varphi = \left\{ \log \sinh\left(\frac{r}{2}\right) - \log \cosh\left(\frac{r}{2}\right) \right\} \Big|_{r_2}^u,$$

thus we derive the C^0 -estimates

$$(7.17) \quad |\varphi| \leq \log c.$$

As in the proof of [7, Lemma 7.5], an upper bound for the principal curvatures of the slices $\{x^0 = \text{const}\}$ intersecting $M(t)$ satisfies

$$(7.18) \quad \bar{\kappa} \leq \frac{\sup \cosh u(0, \cdot)}{\sinh u_{\min}} \leq \frac{c}{u_{\min}},$$

and from [5, (2.7.83)] we infer that the uniform boundedness of v .

$$(7.19) \quad v \leq e^{\bar{\kappa}(u_{\max} - u_{\min})} \leq e^{c(\frac{u_{\max}}{u_{\min}})^{-1}},$$

concluding further that

$$(7.20) \quad |D\varphi|^2 = v^2 - 1 \leq c.$$

Define

$$(7.21) \quad \tilde{g}^{ij} = \sigma^{ij} - v^{-2} \varphi^i \varphi^j,$$

where

$$(7.22) \quad \varphi^i = \sigma^{ik} \varphi_k.$$

Due to the boundedness of v the metrics \tilde{g}_{ij} and σ_{ij} are equivalent, thus we can raise the indices of φ_{ij} by \tilde{g}_{ij} and by employing the relation [6, (3.26)]

$$(7.23) \quad h_j^i = v^{-1} \vartheta^{-1} \{ -(\sigma^{ik} - v^{-2} \varphi^i \varphi^k) \varphi_{jk} + \dot{\vartheta} \delta_j^i \},$$

we infer

$$(7.24) \quad \tilde{g}^{ik} \varphi_{jk} = -v \vartheta h_j^i + \dot{\vartheta} \delta_j^i,$$

concluding further from (7.10)

$$(7.25) \quad \|\varphi_{ij}\|^2 \leq c(v^2 \vartheta^2 |A|^2 + n \dot{\vartheta}^2)$$

is bounded from above for all $t \in [t_1, t_2]$. We choose coordinates such that $\tilde{\Gamma}_{ij}^k$ in a fixed point vanishes. Denote the covariant derivative with respect to σ_{ij} by a colon. In such coordinates

$$(7.26) \quad \Gamma_{ij}^k = \frac{1}{2} g^{km} (g_{mi:j} + g_{mj:i} - g_{ij:m}).$$

From

$$(7.27) \quad g^{ij} = \vartheta^{-2} \tilde{g}^{ij}$$

we compute

$$(7.28) \quad g^{km} g_{mi:j} = \tilde{g}^{km} \{ \varphi_{mj} \varphi_i + \varphi_{ij} \varphi_m + 2 \cosh u \varphi_j (\varphi_m \varphi_i + \sigma_{mi}) \}.$$

Using the estimates for φ proved before, we conclude that $\Gamma_{ij}^k - \tilde{\Gamma}_{ij}^k$ are uniformly bounded independent of t_1 and t_2 . \square

Define a new time parameter as

$$(7.29) \quad \tau = -\log \Theta,$$

then

$$(7.30) \quad \frac{dt}{d\tau} = \Theta \frac{\sinh \Theta}{\cosh \Theta}.$$

In the following we denote the differentiation with respect to t by a dot and differentiation with respect to τ by a prime.

7.4. Lemma. *The rescaled quantity $\tilde{F} = F\Theta$ satisfies the inequality*

$$(7.31) \quad \sup_{M(t_1)} \tilde{F} \leq c \inf_{M(t_2)} \tilde{F}$$

with a uniform constant $c > 0$.

Proof. \tilde{F} satisfies the equation

$$(7.32) \quad \tilde{F}' = \dot{F}\Theta^2 \frac{\sinh \Theta}{\cosh \Theta} - \tilde{F},$$

and from the evolution equation of F in [7, (2.8)] we conclude further

$$(7.33) \quad \tilde{F}' + \tilde{F} - \{F^{ij}F_{;ij} + F^{ij}h_{ik}h_j^k F + K_N F^{ij}g_{ij}F\}\Theta^2 \frac{\sinh \Theta}{\cosh \Theta} = 0.$$

We consider the non-trivial term in (7.33)

$$(7.34) \quad -F^{ij}F_{;ij}\Theta^2 \frac{\sinh \Theta}{\cosh \Theta}.$$

In view of (7.27), the pinching estimate and the boundedness of v , $\Theta^2 F^{ij}$ and σ^{ij} are equivalent and hence uniformly positive definite. Furthermore,

$$(7.35) \quad F_{;ij} = F_{;ij} - \{I_{ij}^k - \tilde{I}_{ij}^k\}F_k.$$

Hence we conclude from Lemma 7.3 that \tilde{F} satisfies a uniform parabolic equation of the form

$$(7.36) \quad \tilde{F}' - a^{ij}\tilde{F}_{;ij} + b^i\tilde{F}_i + c\tilde{F} = 0$$

in the cylinder $[\tau_1, \tau_2] \times \mathbb{S}^n$, where $\tau_i = -\log \Theta(t_i, T^*)$, with uniformly bounded coefficients. The statement follows then from the parabolic Harnack inequality. \square

7.5. Corollary. *The rescaled principal curvatures $\tilde{\kappa}_i = \kappa\Theta$ are uniformly bounded from below.*

Proof. Consider a point (t, ξ) in $M(t)$ such that

$$(7.37) \quad u(t, \xi) = \sup_{M(t)} u.$$

In view of [5, (1.5.10)], it holds in (t, ξ)

$$(7.38) \quad h_{ij} \geq \bar{h}_{ij}, \quad g_{ij} = \bar{g}_{ij}, \quad \kappa_i \geq \bar{\kappa}_i = \frac{\cosh u}{\sinh u},$$

where we denote the quantity of the slices $\{x^0 = \text{const}\}$ with a bar. In view of (7.15)

$$(7.39) \quad \sup_{M(t)} \tilde{F} \geq F(\tilde{\kappa}_i(t, \xi)) \geq F\left(\frac{\cosh u(t, \xi)}{\sinh u(t, \xi)}\Theta(t, T^*)\right) \geq c > 0.$$

The statement follows from the pinching estimates and Lemma 7.4. \square

Let $x_0 \in \mathbb{H}^{n+1}$ be the point the flow hypersurfaces are shrinking to and introduce geodesic polar coordinates around it. Write $M(t) = \text{graph } u(t, \cdot)$ and let

$$(7.40) \quad \tilde{u}(\tau, \xi) = u(t, \xi) \Theta(t, T^*)^{-1},$$

$$(7.41) \quad \tau_\delta = -\log \Theta(t_\delta, T^*), \quad Q(\tau_\delta, \infty) = [\tau_\delta, \infty) \times \mathbb{S}^n.$$

Using the same argument as in [7, Lemma 7.9, Lemma 7.10] we conclude that

7.6. Lemma. *The quantities v and $|D\tilde{u}|$ are uniformly bounded from above and \tilde{u} is uniformly bounded from below and above in $Q(\tau_\delta, \infty)$. \square*

Let

$$(7.42) \quad \varphi = - \int_u^{\Theta(0, T^*)} \vartheta^{-1},$$

then

$$(7.43) \quad \varphi_i = \vartheta^{-1} u_i, \quad \varphi_{ij} = \vartheta^{-1} u_{ij} - \cosh u \vartheta^{-2} u_i u_j,$$

and

$$(7.44) \quad \vartheta^{-2} |D^2 u|^2 + |D\tilde{u}|^4 \cosh^2 u - 2\vartheta^{-1} |D^2 u| |D\tilde{u}|^2 \cosh u \leq |D^2 \varphi|^2.$$

Since $|D^2 \varphi|$ and $|D\tilde{u}|$ are bounded, we conclude that the C^2 -norm of \tilde{u} is uniformly bounded, where the covariant derivatives of \tilde{u} and φ are taken with respect to σ_{ij} . From [5, Remark 1.5.1, Lemma 2.7.6] we conclude that

$$(7.45) \quad \frac{\sinh \Theta}{\cosh \Theta} Fv = \Phi(x, \tau, \tilde{u}, \tilde{u} e^{-\tau}, D\tilde{u}, D^2 \tilde{u}),$$

where Φ is a smooth function with respect to its arguments, and

$$(7.46) \quad \begin{aligned} \Phi^{ij} &\equiv \frac{\partial \Phi}{\partial (-\tilde{u}_{ij})} = F^{ij} \Theta \frac{\sinh \Theta}{\cosh \Theta}, \\ \Phi^{ij, kl} &= F^{ij, kl} \vartheta^{-1} \Theta^2 \frac{\sinh \Theta}{\cosh \Theta}. \end{aligned}$$

Hence by applying first the Krylov and Safonov, then the Schauder estimates, we deduce (cf. [5, Remark 2.6.2])

7.7. Theorem. *The rescaled function \tilde{u} satisfies the uniformly parabolic equation*

$$(7.47) \quad \tilde{u}' = -\Phi + \tilde{u}$$

in $Q(\tau_\delta, \infty)$ and $\tilde{u}(\tau, \cdot)$ satisfies a priori estimates in $C^\infty(\mathbb{S}^n)$ independently of τ .

8. CONVERGENCE TO A SPHERE

The aim of this section is to prove that \tilde{u} converges exponentially fast to the constant function 1 if F is strictly concave or $F = \frac{1}{n}H$. Comparing the proof in [7, Section 8], we should handle a term stemming from the negative curvature of the ambient space $K_N < 0$.

8.1. **Lemma.** *There exists a positive constant C such that*

$$(8.1) \quad F^{kl}g_{kl}|A|^2 - FH \leq C \sum_{i < j} (\kappa_i - \kappa_j)^2.$$

Proof. The proof is similar to [7, Lemma 8.2]. Let

$$(8.2) \quad \varphi = F^{kl}g_{kl}|A|^2 - FH.$$

Denote the partial derivatives of φ with respect to κ_i by φ_i , then

$$(8.3) \quad \varphi_j = \sum_{i=1}^n F_{ij}|A|^2 + \sum_{i=1}^n 2F_i\kappa_j - F_jH - F,$$

$$(8.4) \quad \begin{aligned} \varphi_{jk} &= \sum_{i=1}^n F_{ijk}|A|^2 + \sum_{i=1}^n 2F_{ij}\kappa_k + \sum_{i=1}^n 2F_{ik}\kappa_j \\ &\quad + 2\delta_{jk} \sum_{i=1}^n F_i - F_{jk}H - F_j - F_k. \end{aligned}$$

Therefore

$$(8.5) \quad \varphi(\kappa_n, \dots, \kappa_n) = 0, \quad \varphi_j(\kappa_n, \dots, \kappa_n) = 0 \quad \forall j = 1, \dots, n.$$

by using the Euler's homogeneous relation and the normalization (1.6). Furthermore, φ_{jk} are uniformly bounded from above, since φ_{jk} are homogeneous of grad 0 and $\frac{\kappa_i}{|A|}$ are compactly contained in the defining cone. The statement follows by an argument using Taylor's expansion up to the second order similar to those in [7, Lemma 8.2]. \square

We want to estimate the function

$$(8.6) \quad f_\sigma = F^{-\alpha}(|A|^2 - nF^2),$$

where

$$(8.7) \quad \alpha = 2 - \sigma,$$

and $0 < \sigma < 1$ small. For simplicity we drop the subscript σ of f_σ . In the following we always assume that F satisfies the assumption 1.1.

By Lemma 8.1 we have the following inequality corresponding to [7, Lemma 8.3].

8.2. Lemma. *Let F be strictly concave, then there exist uniform constants $\epsilon > 0$ and $C > 0$, such that*

$$(8.8) \quad \begin{aligned} -F^{ij}f_{ij} + 2\epsilon^2 F^{ij}h_{ki}h_j^k f &\leq \alpha F^{-1}F^{ij}F_{;ij}f + 2(\alpha - 1)F^{-1}F^{ij}F_i f_j \\ &\quad - 2\{h^{ij} - FnF^{ij}\}F^{-\alpha}F_{;ij} - 2\epsilon^2|DA|^2F^{-\alpha} + 2Cf. \end{aligned}$$

Corresponding to [7, Lemma 8.5] we have

8.3. Lemma. *Let F be strictly concave, then there exist positive constants C and c such that for any $p \geq 2$, any $\delta > 0$ and any $0 \leq t < T^*$*

$$(8.9) \quad \begin{aligned} \epsilon^2 \int_M F^{ij}h_{ki}h_j^k f^p &\leq \{\delta^{-1}c(p-1) + c\} \int_M F^{ij}f_i f_j f^{p-2} \\ &\quad + \{\delta c(p-1) + c\} \int_M |DA|^2 F^{-\alpha} f^{p-1} + 2C \int_M f^p. \end{aligned}$$

Parallel to [7, Lemma 8.6] we have

8.4. Lemma. *Let F be strictly concave, then there exist $C_1 > 0$ and $\sigma_0 > 0$ such that for all*

$$(8.10) \quad p \geq 4c\epsilon^{-2}, \quad \sigma \leq \min(\frac{1}{4}c^{-1}\epsilon^3 p^{-1/2}, \sigma_0),$$

the estimate

$$(8.11) \quad \|f\|_{p,M} \leq C_1 \quad \forall t \in [0, T^*)$$

holds, where $C_1 = C_1(M_0, p)$ and $\sigma_0 = \sigma_0(F, M_0)$.

Proof. Multiply [7, (8.30)] with pf^{p-1} and integrate by parts, and note that

$$(8.12) \quad d\mu_t = \mu_t dx \quad \text{on } M_t,$$

where

$$(8.13) \quad \frac{d}{dt}\mu_t = \frac{d}{dt}\sqrt{g} = \frac{1}{2}\mu_t g^{ij}\dot{g}_{ij} = -FH\mu_t,$$

thus

$$(8.14) \quad \frac{d}{dt} \int_M f^p = p \int_M f^{p-1} f' - \int_M HF f^p,$$

and

$$(8.15) \quad \begin{aligned} \frac{d}{dt} \int_M f^p + \frac{1}{2}p(p-1) \int_M F^{ij}f_i f_j f^{p-2} + \epsilon^2 p \int_M |DA|^2 F^{-\alpha} f^{p-1} \\ \leq \sigma p \int_M F^{ij}h_{ki}h_j^k f^p + 4Cp \int_M f^p. \end{aligned}$$

By choosing

$$(8.16) \quad c_0 = \frac{1}{4}c, \quad \sigma \leq \min(\epsilon^3 p^{-1/2} c_0^{-1}, \sigma_0), \quad \delta = \epsilon p^{-1/2},$$

and by using (8.9), the right-hand side of inequality (8.15) can be estimated from above by

(8.17)

$$\begin{aligned}
& \epsilon p^{1/2} c_0^{-1} \{ \epsilon^2 \int_M F^{ij} h_{ki} h_j^k f^p \} + 4Cp \int_M f^p \\
& \leq \epsilon p^{1/2} c_0^{-1} \{ \delta^{-1} c(p-1) + c \} \int_M F^{ij} f_i f_j f^{p-2} \\
& \quad + \epsilon p^{1/2} c_0^{-1} \{ \delta c(p-1) + c \} \int_M |DA|^2 F^{-\alpha} f^{p-1} + \{ 2C\epsilon p^{1/2} c_0^{-1} + 4Cp \} \int_M f^p \\
& = c_0^{-1} \{ p(p-1)c + \epsilon p^{1/2} c \} \int_M F^{ij} f_i f_j f^{p-2} \\
& \quad + c_0^{-1} \{ \epsilon^2(p-1)c + \epsilon p^{1/2} c \} \int_M |DA|^2 F^{-\alpha} f^{p-1} + \{ 2C\epsilon p^{1/2} c_0^{-1} + 4Cp \} \int_M f^p \\
& \leq \frac{1}{2} p(p-1) \int_M F^{ij} f_i f_j f^{p-2} + \frac{1}{2} \epsilon^2(p-1) \int_M |DA|^2 F^{-\alpha} f^{p-1} + 5Cp \int_M f^p.
\end{aligned}$$

From (8.15), (8.17) we conclude that

$$(8.18) \quad \frac{d}{dt} \int_M f^p \leq 5Cp \int_M f^p,$$

and the Gronwall's lemma leads to

$$(8.19) \quad \int_M f^p \leq \int_M f^p|_{t=0} \cdot \exp(5CpT^*),$$

$$(8.20) \quad \|f\|_p = \left(\int_M f^p \right)^{\frac{1}{p}} \leq e^{5CT^*} (|M_0| + 1) \sup_{0 \leq \sigma \leq 1/2} \sup_{M_0} f_\sigma.$$

□

To proceed further, we use the Stampacchia iteration scheme as in the Huisken's paper [10, Theorem 5.1], as well as [11, Theorem 5.1]. Note that \mathbb{H}^{n+1} is simply connected and has constant sectional curvature $K_N = -1$, thus the Sobolev inequality in [9, Theorem 2.1] has the form

8.5. Lemma. *Let v be a nonnegative Lipschitz function on M , then there exists a constant $c = c(n) > 0$, such that*

$$(8.21) \quad \left(\int_M |v|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \leq c \{ \int_M |Dv| + \int_M H|v| \}.$$

Corresponding to [7, Theorem 8.7], we have

8.6. Theorem. *Let F be strictly concave or $F = \frac{1}{n}H$, then there exist constants $\delta > 0$ and $c_0 > 0$, such that*

$$(8.22) \quad |A|^2 - nF^2 \leq c_0 F^{2-\delta}.$$

Proof. As in the proof of [10, Theorem 5.1] let $f_{\sigma,k} = \max(f_\sigma - k, 0)$ for all $k \geq k_0 = \sup_{M_0} f_\sigma$ and denote by $A(k)$ the set where $f_\sigma > k$. We obtain with $v = f_{\sigma,k}^{p/2}$ for $p \geq 4c\epsilon^{-2}$,

$$(8.23) \quad \begin{aligned} \frac{d}{dt} \int_{A(k)} v^2 + \int_{A(k)} |Dv|^2 &\leq \sigma p \int_{A(k)} H^2 f_\sigma^p + 5Cp \int_{A(k)} f_\sigma^p \\ &\leq C(p) \int_{A(k)} H^2 f_\sigma^p. \end{aligned}$$

By applying Lemma 8.5 we can bound f_σ for σ small as in the proof of [10, Theorem 5.1]. The case $F = \frac{1}{n}H$ is proved in [11, Lemma 5.1]. \square

8.7. Lemma. *Let F be strictly concave or $F = \frac{1}{n}H$ and $\tilde{M}(\tau)$ be the rescaled hypersurfaces, then there are constants $c, \delta > 0$ such that*

$$(8.24) \quad \int_{\tilde{M}} |D\tilde{A}|^2 \leq ce^{-\delta\tau} \quad \forall \tau_0 \leq \tau < \infty,$$

where

$$(8.25) \quad \tau_0 = -\log \Theta(0, T^*), \quad |D\tilde{A}|^2 = \Theta^2 g^{ij} h_{l;i}^k \Theta h_{k;j}^l \Theta.$$

Proof. Choose

$$(8.26) \quad f = F^{-2} \{|A|^2 - nF^2\}.$$

From Theorem 8.6 we infer

$$(8.27) \quad f \leq c_0 F^{-\delta} \leq c \Theta^\delta = ce^{-\delta\tau} \quad \forall \tau \geq \tau_0,$$

and from Theorem 7.7 we obtain

$$(8.28) \quad |D^m A| \leq c|A| \quad \forall m \geq 1.$$

Integrating inequality (8.8) over M , using integration by parts and using relation (8.28), we infer

$$(8.29) \quad 2\epsilon^2 \int_M |DA|^2 F^{-2} \leq c \int_M f.$$

Hence (8.24) follows by rescaling (8.29). \square

Using the same proof of [7, Lemma 8.10] we have

8.8. Lemma. *There are positive constants c and δ such that for all $\tau \geq \tau_0$*

$$(8.30) \quad \tilde{F}_{\max} - \tilde{F}_{\min} \leq ce^{-\delta\tau},$$

and

$$(8.31) \quad \|D\tilde{F}\| \leq ce^{-\delta\tau}.$$

\square

8.9. Lemma. *There are positive constants c and δ such that for all $\tau \geq \tau_0$*

$$(8.32) \quad |D\tilde{u}| \leq ce^{-\delta\tau},$$

where

$$(8.33) \quad |D\tilde{u}|^2 = \sigma^{ij} \tilde{u}_i \tilde{u}_j.$$

Proof. As in the proof of [7, Lemma 8.12], we let

$$(8.34) \quad \varphi = \log \tilde{u}, \quad w = \frac{1}{2}|D\varphi|^2,$$

then

$$(8.35) \quad \varphi' = -e^{-\varphi} \tilde{F} \Theta^{-1} \frac{\sinh \Theta}{\cosh \Theta} v + 1.$$

Differentiate now (8.35) with respect to $\varphi^k D_k$ we obtain

$$(8.36) \quad \begin{aligned} w' = & 2e^{-\varphi} w \tilde{F} \Theta^{-1} \frac{\sinh \Theta}{\cosh \Theta} v - e^{-\varphi} \tilde{F} \Theta^{-1} \frac{\sinh \Theta}{\cosh \Theta} v^{-1} \sinh^{-2} u u^2 w_k \varphi^k \\ & + R_1 + R_2, \end{aligned}$$

where

$$(8.37) \quad \begin{aligned} R_1 &= -e^{-\varphi} \frac{\sinh \Theta}{\cosh \Theta} v F_k \varphi^k, \\ R_2 &= e^{-\varphi} \tilde{F} \frac{\sinh \Theta}{\Theta \cosh \Theta} v^{-1} |D\varphi|^4 \sinh^{-3} u \{u^3 \cosh u - u^2 \sinh u\} \geq 0. \end{aligned}$$

In view of (8.31) R_1 decays exponentially. Thus the function

$$(8.38) \quad w_{\max} = \sup_{\tilde{M}(\tau)} w$$

is Lipschitz and satisfies

$$(8.39) \quad w'_{\max} \geq 2e^{-\varphi} w \tilde{F} \Theta^{-1} \frac{\sinh \Theta}{\cosh \Theta} v - ce^{-\delta\tau}$$

for almost every $\tau \geq \tau_0$. Using the same argument as in [7, Lemma 8.12] we conclude that

$$(8.40) \quad w_{\max}(\tau) \leq \frac{c}{\delta} e^{-\delta\tau} \quad \forall \tau \geq \tau_0.$$

□

The same arguments of [7, Corollary 8.13] and the interpolation inequalities for the C^m -norms (cf. [6, Corollary 6.2]) yield

8.10. Theorem. *Let F be strictly concave or $F = \frac{1}{n}H$, then the rescaled function \tilde{u} converges in $C^\infty(\mathbb{S}^n)$ to the constant function 1 exponentially fast.*

□

8.11. Lemma. *Let F be strictly concave or $F = \frac{1}{n}H$, then there exist positive constants c and δ such that*

$$(8.41) \quad |\tilde{F}(\tau, \cdot) - 1| \leq ce^{-\delta\tau} \quad \forall \tau \geq \tau_0.$$

Proof. Observe that for τ_1 sufficiently large we have

$$(8.42) \quad \left| \frac{\sinh \Theta}{\cosh \Theta} - \Theta \right| \leq c\Theta^2 \quad \forall \tau \geq \tau_1.$$

The rest of the proof is identical to [7, Lemma 8.16]. \square

9. INVERSE CURVATURE FLOWS

Let $M(t)$ be the flow hypersurfaces of the direct flow in \mathbb{H}^{n+1} and write $M(t)$ as graphs $M(t) = \text{graph } u(t, \cdot)$ with respect to the geodesic polar coordinates centered in the point where the direct flow shrinks to. By applying an isometry we may assume that the point x_0 is the Beltrami point. The polar hypersurfaces $M(t)^*$ are the flow hypersurfaces of the corresponding inverse curvature flow in the de Sitter space. Write $M(t)^* = \text{graph } u^*(t, \cdot)$ over \mathbb{S}^n .

9.1. Lemma. *The functions u, u^* satisfy the relations*

$$(9.1) \quad u_{\max} = -u_{\min}^* \quad \forall t \in [t_\delta, T^*),$$

$$(9.2) \quad u_{\min} = -u_{\max}^* \quad \forall t \in [t_\delta, T^*).$$

Proof. We use the relation [5, (10.4.65)]

$$(9.3) \quad \tilde{x}^0 = \frac{r}{\sqrt{1-r^2}},$$

and note that by comparing [5, (10.2.5)] and the metric in the eigentime coordinate system in N (2.13) we infer that

$$(9.4) \quad \cosh^2 u^* = 1 + |\tilde{x}^0|^2.$$

From (6.8) we infer that

$$(9.5) \quad r = \tanh u.$$

Since we have switched the light cone such that the uniformly convex slices are contained in $\{\tau < 0\}$, we deduce that

$$(9.6) \quad u^* = -\text{arcsinh}(\tilde{v} \sinh u) = -\text{arcsinh} \tilde{\chi}.$$

In a point where u^* attains its minimum, there holds $v = 1$ in view of Lemma 7.1. Thus $u = -u^*$ and u attains its maximum in such a point. This proves (9.1). The proof of (9.2) is similar. \square

9.2. Corollary. *There exists a positive constant c such that*

$$(9.7) \quad -c \leq w \equiv u^* \Theta^{-1} \leq -c^{-1} \quad \forall t \in [t_\delta, T^*).$$

\square

Define $\vartheta(u) = \cosh(u)$ and $\bar{g}_{ij} = \vartheta^2 \sigma_{ij}$. We prove in the following that w is uniformly bounded in $C^\infty(\mathbb{S}^n)$. For simplicity, we write in the following u instead u^* for the graphs of the flow hypersurfaces in the de Sitter space. The proof of C^1 -estimates of w is similar to [5, Theorem 2.7.11].

9.3. Lemma. *There exists a positive constant c such that*

$$(9.8) \quad |Dw|^2 \equiv \sigma^{ij} w_i w_j \leq c \quad \forall t \in [t_\delta, T^*).$$

Proof. Since

$$(9.9) \quad \|Du\|^2 \equiv g^{ij} u_i u_j = v^{-2} \bar{g}^{ij} u_i u_j \equiv v^{-2} |Du|^2,$$

we first estimate $\|Du\|\Theta^{-1}$. Let λ be a real parameter to be specified later and define

$$(9.10) \quad G = \frac{1}{2} \log(\|Du\|^2 \Theta^{-2}) + \lambda u \Theta^{-1}.$$

There is $x_0 \in \mathbb{S}^n$ such that

$$(9.11) \quad G(x_0) = \sup_{\mathbb{S}^n} G,$$

and thus in x_0

$$(9.12) \quad 0 = G_i = \|Du\|^{-2} u_{ij} u^j + \lambda u_i \Theta^{-1},$$

where the covariant derivatives are taken with respect to g_{ij} and

$$(9.13) \quad u^i = g^{ij} u_j = v^{-2} \bar{g}^{ij} u_j.$$

Since

$$(9.14) \quad h_{ij} v^{-1} = -u_{ij} - \dot{\vartheta} \sigma_{ij},$$

we infer that

$$(9.15) \quad \begin{aligned} \lambda \|Du\|^{-4} \Theta^{-4} &= -u_{ij} u^i u^j \Theta^{-3} \\ &= v^{-1} h_{ij} u^i u^j \Theta^{-3} + \dot{\vartheta} \sigma_{ij} u^i u^j \Theta^{-3}. \end{aligned}$$

By considering the dual flow in the hyperbolic space, we conclude that $h_{ij} > 0$. Furthermore,

$$(9.16) \quad \dot{\vartheta} \sigma_{ij} u^i u^j \Theta^{-3} = (\dot{\vartheta} \Theta^{-1}) \vartheta^{-1} v^{-2} \|Du\|^2 \Theta^{-2}.$$

By applying [5, Theorem 2.7.11] directly, we conclude that v^{-2} is uniformly bounded. Note $\dot{\vartheta} \Theta^{-1} \leq c$. Let c_0 be an upper bound for $(\dot{\vartheta} \Theta^{-1}) \vartheta^{-1} v^{-2}$ and by choosing $\lambda < -c_0$ we conclude that $\|Du\|\Theta^{-1}$ can not be too large in x_0 . Thus $\|Du\|\Theta^{-1}$ is uniformly bounded from above. We conclude that

$$(9.17) \quad \sigma^{ij} w_i w_j = \|Du\|^2 \Theta^{-2} \theta^2 v^2$$

is uniformly bounded. \square

9.4. Lemma. *There exists a positive constant c such that for all $m \geq 2$*

$$(9.18) \quad |D^m w|^2 \leq c \quad \forall t \in [t_\delta, T^*).$$

Proof. Let $(\hat{h}^{ij}) = (h_{ij})^{-1}$ be the inverse of the second fundamental form in \mathbb{H}^{n+1} and \tilde{h}_{ij} the second fundamental form in N . We consider the mixed tensor

$$(9.19) \quad \hat{h}_i^j = g_{ik} \hat{h}^{kj}, \quad \tilde{h}_i^j = \tilde{g}^{kj} \tilde{h}_{ki},$$

where g_{ij} and $\tilde{g}_{ij} = h_i^k h_{kj}$ are the metrics of hypersurfaces in \mathbb{H}^{n+1} resp. N . From the relation

$$(9.20) \quad \tilde{\kappa}_i = \kappa_i^{-1},$$

we infer that

$$(9.21) \quad \tilde{h}_i^j = \hat{h}_i^j.$$

From Theorem 7.7 we infer that $h_i^j \Theta$ are uniformly bounded in $C^\infty(\mathbb{S}^n)$ and due to Lemma 7.2 and Corollary 7.5 there are constants $c_1, c_2 > 0$ such that

$$(9.22) \quad 0 < c_1 \delta_i^j \leq h_i^j \Theta \leq c_2 \delta_i^j,$$

and thus $\tilde{h}_i^j \Theta^{-1} = \hat{h}_i^j \Theta^{-1}$, as the inverse of $h_i^j \Theta$, are uniformly bounded in $C^\infty(\mathbb{S}^n)$. We switch now our notation by considering the quantities in N without writing a tilde. Denote the covariant derivatives with respect to \bar{g}_{ij} resp. σ_{ij} by a semicolon resp. a colon. In view of [5, Remark 1.6.1, Lemma 2.7.6] we have

$$(9.23) \quad \begin{aligned} v^{-1} h_{ij} &= -v^{-2} u_{;ij} - \dot{\vartheta} \vartheta \sigma_{ij} \\ &= -v^{-2} \{ u_{;ij} - \frac{1}{2} \bar{g}^{km} ((\vartheta^2)_j \sigma_{mi} + (\vartheta^2)_i \sigma_{mj} - (\vartheta^2)_m \sigma_{ij}) u_k \} - \dot{\vartheta} \vartheta \sigma_{ij}. \end{aligned}$$

Therefore,

$$(9.24) \quad u_{;ij} = -v h_{ij} + 2\vartheta^{-1} \dot{\vartheta} u_i u_j - \vartheta \dot{\vartheta} \sigma_{ij}.$$

By considering the dual flow in hyperbolic space, we infer that

$$(9.25) \quad |A| \Theta^{-1} \leq c,$$

and note that

$$(9.26) \quad \bar{g}^{ij} \leq \bar{g}^{ij} + v^{-2} \check{u}^i \check{u}^j = g^{ij},$$

where

$$(9.27) \quad \check{u}^i = \bar{g}^{ij} u_j,$$

we conclude that

$$(9.28) \quad \sigma^{ik} \sigma^{jl} h_{ij} h_{kl} \leq c |A|^2.$$

In view of $\dot{\vartheta} \Theta^{-1} \leq c$ we conclude that $|D^2 w|^2$ is uniformly bounded. Contract (9.24) with g^{ij} we conclude further

$$(9.29) \quad -g^{ij} w_{;ij} - \vartheta^{-3} \dot{\vartheta} \Theta v^{-2} |Dw|^2 + v H \Theta^{-1} + n \vartheta^{-1} \dot{\vartheta} \Theta^{-1} = 0.$$

Since v is uniformly bounded, (9.29) is a uniformly elliptic equation in w with bounded coefficients. A bootstrapping procedure with Schauder theory yields for all $m \in \mathbb{N}$

$$(9.30) \quad |w|_{m, \mathbb{S}^n} \leq c_m \quad \forall t \in [0, T^*].$$

□

From Lemma 8.10 and preceding results in Section 9 we conclude

9.5. Theorem. *Let the geodesic polar coordinates (τ, ξ^i) of N be specified in Section 2. Represent the inverse curvature flow (1.5) in N as graphs over \mathbb{S}^n , $M(t)^* = \text{graph } u^*(t, \cdot)$, where the curvature function \tilde{F} satisfies the assumption 1.1. Then u^* converges to the constant function 0 in $C^\infty(\mathbb{S}^n)$. The rescaled function $w = u^* \Theta^{-1}$ are uniformly bounded in $C^\infty(\mathbb{S}^n)$. When the curvature function F of the corresponding contracting flow is strictly concave or $F = \frac{1}{n}H$, then $w(\tau, \cdot)$ converges in $C^\infty(\mathbb{S}^n)$ to the constant function -1 exponentially fast.* □

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